CALCULUS OF VARIATION

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Find the shortest distance between the point A(1,0) and the ellipse $4x^2+9y^2=36$

- ***Solution:***

If S is the length of the arc of the curve y=f(x) connecting the point (x_0, y_0) and(x_1, y_1).

Then S=$\int_{x_0}^{x_1} \sqrt{1+y'^2}$ dx

To find the minimum value of the functional S, when the left end (x_0, y_0) is fixed at (1,0) and the right end (x_1, y_1) moves along $4\boldsymbol{x}^2+9\boldsymbol{y}^2=36$

To find the extremum value of the integral

I=$\int_{x_0}^{x_1} \sqrt{1+y'^2}$dx

If I is minimum, then $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$, where F=$\sqrt{1+y'^2}$

$-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)=0$

$\Rightarrow\frac{\boldsymbol{y}'}{\sqrt{\boldsymbol{1}+\boldsymbol{y}'^2}}=$k $\Rightarrow y'^2=\frac{k^2}{1-k^2}$

$\therefore y' = \frac{k}{\sqrt{1-k^2}} = c_1$

$y = c_1 x + c_2 \qquad \rightarrow ①$

Also given the left end (1,0)

$① \Rightarrow c_1 + c_2 = 0 \rightarrow ②$

Right end (x_1, y_1) satisfies ① and the curve eqn

$4\boldsymbol{x^2} + 9\boldsymbol{y^2} = 36$

$\Rightarrow y_1 = c_1 x_1 + c_2 \rightarrow ③$

$4\boldsymbol{x^2}_1 + 9\boldsymbol{y^2}_1 = 36 \quad \Rightarrow y_1 = \frac{2}{3}\sqrt{(9 - \boldsymbol{x^2}_1)}$

$\therefore \Phi = y_1 = \frac{2}{3}\sqrt{(9 - \boldsymbol{x^2}_1)}$

The transversality condition at (x_1, y_1) we have

$\left[F + (\Phi' - y')\ \frac{\partial F}{\partial y'}\right]_{(x_1, y_1)}$ =0,then $\Phi' = \frac{-2x}{3\sqrt{(9 - \boldsymbol{x^2})}}$

- $$\left[\sqrt{1+y'^2}+\left(\frac{-2x}{3\sqrt{(9-\boldsymbol{x^2}_1)}}-y'\right)\frac{y'}{\sqrt{1+y'^2}}\right]_{(x_1,y_1)}=0$$
- $$\frac{-2\,x_1 y'}{3\sqrt{(9-\boldsymbol{x^2}_1)}}=-1$$

$$y'_1=\frac{3\sqrt{9-\boldsymbol{x^2}_1}}{2x_1}\rightarrow④$$

From ③ $y'_1=c_1$

$$\therefore c_1=\frac{3\sqrt{9-\boldsymbol{x^2}_1}}{2x_1}\rightarrow⑤$$

From ② $c_2=-c_1$

$$y_1=c_1x_1+c_2$$

$$\frac{2}{3}\sqrt{(9-\boldsymbol{x^2}_1)}=c_1x_1-c_1$$

$$=c_1(x_1-1)$$

Sub value of c_1

$$\frac{2}{3}\sqrt{(9-\boldsymbol{x^2}_1)}=\frac{3\sqrt{9-\boldsymbol{x^2}_1}}{2x_1}(x_1-1)$$

$$\Rightarrow x_1=\frac{9}{5}$$

$$\therefore c_1=\frac{3\sqrt{9-\frac{81}{25}}}{2\frac{9}{5}}$$

$$\Rightarrow c_1=2$$

$\therefore$ The least value of $S=\int_{x_0}^{x_1}\sqrt{1+y'^2}dx$

$$=\int_1^{\frac{9}{5}}\sqrt{1+4}dx$$

$$S=\sqrt{5}\left[\frac{9}{5}-1\right]=\frac{4}{\sqrt{5}}$$